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Fractional Zero Curvature Equation and Generalized Hamiltonian Structure of Soliton Equation Hierarchy

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Abstract We presented the fractional zero curvature equation and generalized Hamiltonian structure by using of the differential forms of fractional orders. Example of the fractional AKNS soliton equation hierarchy and its Hamiltonian system are obtained.

Keywords Fractional zero curvature equation · Generalized Hamiltonian structure · Fractional AKNS equation hierarchy

1 Introduction

The theory of integrals and derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald, and Letnikov. Derivatives and integrals of fractional order [1, 2] have found many applications in recent studies in physics. The interest in fractional analysis has been growing continually during the past few years. Fractional analysis has numerous applications: kinetic theories [3, 4, 9], statistical mechanics [10, 11], dynamics in complex media [12, 13], and many others [5–8]. In the past few decades many authors have pointed out that fractional-order models are more appropriate than integer-order models for various real materials. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models in which such effects are, in fact, neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. How to obtain the fractional soliton equation and its Hamiltonian structure is an important work. In this paper we present the zero curvature equation and Hamiltonian structure of the fractional soliton

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equation hierarchy by using of differential forms and exterior derivatives of fractional orders [14–16].

In Sect. 2, some information of the fractional differential forms is considered to fix notation. In Sect. 3, the fractional zero curvature equation is presented. In Sect. 4, we consider the Hamiltonian system. In Sect. 5, a fractional Hamilton system is constructed. We discuss an example of the fractional AKNS equation hierarchy and its Hamiltonian system in Sect. 6.

2 Brief Review of Fractional Derivatives and Integrals

The derivatives of arbitrary real order p can be considered as an interpolation of this sequence of operators. We will use for it the notion suggested and used by Davis [17], namely

$$ {}_a\mathbf{D}_t^p f(t), \tag{1}$$

the short name for derivatives of arbitrary order is fractional derivatives.

The subscripts a and t denote the two limits related to the operation of fractional differentiation. Following Ross [18], we will call them the terminals of fractional differentiation. The appearance of the terminals in the symbol of fractional is essential. This helps to avoid ambiguities in applications of fractional derivatives to real problems. Integrals of arbitrary order $p > 0$

$$ {}_a\mathbf{D}_t^{-p} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+k+1)} \int_a^t (t-\tau)^{p+m} f^{(m+1)}(\tau) d\tau, \tag{2}$$

the formula (2) immediately provides us with the asymptotic of ${}_a\mathbf{D}_t^{-p} f(t)$ at $t = 0$. Derivatives of arbitrary order

$$\begin{aligned} {}_a\mathbf{D}_t^p f(t) &= \lim_{h\to\infty, nh\to t-a} f_n^{(p)}(t) \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+k+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau, \end{aligned} \tag{3}$$

the formula (3) has been obtained under the assumption that the derivatives $f^{(k)}(t)$ ($k = 1, 2, \dots, m+1$) are continuous in closed interval $[a, t]$ and that m is an integer number satisfying the condition $m > p + 1$. The smallest probable value for m is determined by the inequality $m < p < m + 1$.

From the pure mathematical point of view of such a class of functions is narrow. However this class of functions is very important for applications. Because the character of the majority of dynamical processes is smooth enough and does not allow discontinuities. Understanding this fact is important for the proper use of the methods of the fractional calculus in applications, especially because of the fact the Riemann–Liouville definition

$$ {}_a\mathbf{D}_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau) d\tau \quad (m \le p \le m+1) \tag{4}$$

provides an excellent opportunity to weaken the conditions on the function $f(t)$.

We will think how the Riemann–Liouville definition (4) appears as the result of the unification of the notions of integer-order integration and differentiation. Let us suppose that the function $f(\tau)$ is continuous and integrable in every finite interval (a,t); the function $f(t)$ may have on integrable singularity of order $r<1$ at the point $\tau=a$

$$\lim_{\tau\to a}(\tau-a)^r f(t)=\text{const}(\neq 0).$$

Then the integral

$$f^{-1}(t)=\int_a^t f(\tau)d\tau$$

exists and has a finite value, namely equal to 0, as $t\to a$.

In the general case we have the Cauchy formula

$$f^{-n}(t)=\frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau. \tag{5}$$

Let us now suppose that $n\geq 1$ is fixed and take integer $k\geq 0$. Obviously, we will obtain

$$f^{-k-n}(t)=\frac{1}{\Gamma(n)}\mathbf{D}^{-k}\int_a^k (t-\tau)^{n-1}f(\tau)d\tau,$$
$$f^{k-n}(t)=\frac{1}{\Gamma(n)}\mathbf{D}^{k}\int_a^k (t-\tau)^{n-1}f(\tau)d\tau,$$

where the symbol $\mathbf{D}^{-k}(k\geq 0)$ and $\mathbf{D}^{k}(k\geq 0)$ denote k interacted differentiations.

To extend the notion if n-fold integration to non-integer values of n, we can start with the Cauchy formula (5) and replace the integer n in it by a real $p>0$,

$${}_a\mathbf{D}_t^{-p}f(t)=\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}f(\tau)d\tau. \tag{6}$$

In (5) the integer n must satisfy the condition $n\geq 1$; the responding condition for p is weak; for the existence of integral (6) we must have $p>0$. Moreover, under certain reasonable assumptions

$$\lim_{p\to 0}{}_a\mathbf{D}_t^{-p}f(t)=f(t). \tag{7}$$

So we can put ${}_a\mathbf{D}_t^{0}f(t)=f(t)$. If $f(t)$ is continuous for $t\geq a$, the integration of arbitrary real order defined by (6) has the follow important property

$${}_a\mathbf{D}_t^{-p}({}_a\mathbf{D}_t^{-q})f(t)={}_a\mathbf{D}_t^{-p-q}f(t). \tag{8}$$

Indeed, we have

$$\begin{aligned}{}_a\mathbf{D}_t^{-p}({}_a\mathbf{D}_t^{-q})f(t)&=\frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}\,{}_a\mathbf{D}_t^{-p}f(\tau)d\tau\\&=\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t (t-\tau)^{q-1}d\tau\int_a^t (t-\xi)^{p-1}f(\xi)d\xi\\&=\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)d\xi\int_a^t (t-\tau)^{q-1}(t-\xi)^{p-1}d\tau\end{aligned}$$

$$= \frac{1}{\Gamma(p+q)} \int_a^t (t-\xi)^{p+q-1} f(\xi) d\xi$$
$$= {}_a\mathbf{D}_t^{-p-q} f(t).$$

Obviously, we can interchange p and q, so we have

$$ {}_a\mathbf{D}_t^{-p}({}_a\mathbf{D}_t^{-q}) f(t) = {}_a \mathbf{D}_t^{-q}({}_a\mathbf{D}_t^{-p}) f(t) = {}_a \mathbf{D}_t^{-p-q} f(t). \tag{9}$$

Let us consider some properties of the Riemann–Liouville fractional derivatives. The first and maybe the most important property of the Riemann–Liouville fractional derivative is that for $p > 0$ and $t > a$

$$ {}_a\mathbf{D}_t^{p}({}_a\mathbf{D}_t^{-p} f(t)) = f(t), \tag{10}$$
$$ {}_a\mathbf{D}_t^{-p}({}_a\mathbf{D}_t^{p} f(t)) \neq f(t), \tag{11}$$
$$ {}_a\mathbf{D}_t^{n}({}_a\mathbf{D}_t^{p} f(t)) = {}_a \mathbf{D}_t^{n+p} f(t), \tag{12}$$

which means that the Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operation operator of the same order p.

The product rule is

$$\mathbf{D}_t^p(f(t)g(t)) = \sum_{j=0}^{\infty} \binom{p}{j} \mathbf{D}_t^{p-j} f(t) \mathbf{D}_t^j g(t). \tag{13}$$

3 Fractional Zero Curvature Equation

Consider an isospectral Lax pair

$$\begin{cases} \mathbf{D}_x^\alpha \psi = U\psi, \\ \mathbf{D}_t^\alpha = V\psi, \end{cases} \tag{14}$$

where α is arbitrary real number.

From the compatibility condition $\mathbf{D}_t^\alpha \mathbf{D}_x^\alpha \psi = \mathbf{D}_x^\alpha \mathbf{D}_t^\alpha \psi$, we have

$$\mathbf{D}_t^\alpha \mathbf{D}_x^\alpha \psi = \sum_{j=0}^{\infty} \binom{\alpha}{j} \mathbf{D}_t^{\alpha-j} U \mathbf{D}_t^j \psi, \tag{15}$$

$$\mathbf{D}_x^\alpha \mathbf{D}_t^\alpha \psi = \sum_{j=0}^{\infty} \binom{\alpha}{j} \mathbf{D}_x^{\alpha-j} V \mathbf{D}_x^j \psi, \tag{16}$$

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} \mathbf{D}_t^{\alpha-j} U \mathbf{D}_t^j \psi = \sum_{j=0}^{\infty} \binom{\alpha}{j} \mathbf{D}_x^{\alpha-j} V \mathbf{D}_x^j \psi. \tag{17}$$

We choose the terms of $j = 0$ and $j = \alpha$, get the following forum

$$\mathbf{D}_t^\alpha U - \mathbf{D}_x^\alpha V + [U, V] = 0, \tag{18}$$

it is called the fractional zero curvature equation. The nonlinear evolution hierarchy derived from (14) are known as generalized integrable equation.

We consider the following isospectral matrix spectral problem

$$\phi_x = U(u,\lambda)\phi,$$
$$U(u,\lambda) = \begin{pmatrix} -u_3\lambda & u_1 \\ u_2 & u_3\lambda \end{pmatrix} = U_0\lambda + U_1, \qquad \frac{\partial U_0}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda} = 0, \qquad \mathbf{D}_t^{\alpha}\lambda = 0, \tag{19}$$

where λ is a spectral parameter. Because U_0 has multiple eigenvalues, To derive an associated soliton hierarchy, we first solve the adjoint equation

$$\mathbf{D}_x^{\alpha} W = [U, W]$$

of the spectral problem (38) through the generalized Tu scheme [20]. We assume that a solution W is given by

$$W = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{20}$$

Then we have

$$[U, W] = \begin{pmatrix} u_1 A - u_2 B & -2\lambda u_3 B - 2u_1 A \\ 2\lambda u_3 C + 2u_2 A & u_2 B - u_1 A \end{pmatrix}. \tag{21}$$

Take (19), (20) and (21) int the compatibility condition, i.e., the generalized zero curvature equation

$$\mathbf{D}_{t_n}^{\alpha} U - \mathbf{D}_x^{\alpha} W^{(n)} + [U, W^{(n)}] = 0, \tag{22}$$

comparing the coefficient of the λ in (22), leads to a system of evolution equation hierarchy

$$\begin{cases} \mathbf{D}_{t_n}^{\alpha} u_1 - \mathbf{D}_x^{\alpha} B^{(n)} - 2u_1 A^{(n)} = 0, \\ \mathbf{D}_{t_n}^{\alpha} u_2 - \mathbf{D}_x^{\alpha} C^{(n)} + 2u_2 A^{(n)} = 0, \end{cases} \tag{23}$$

where

$$W = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} \begin{pmatrix} A^{(k)} & B^{(k)} \\ C^{(k)} & -A^{(k)} \end{pmatrix} \lambda^{-k},$$

which is the generalized fractional version of the new nonlinear equation hierarchy.

4 Hamiltonian System

In this section, some information of Hamiltonian systems is introduced. Let us consider the symplectic space $(R^{2n}, dp \wedge dq)$. The regular coordinates of $M = R^{2n}$ is $x = (p, q)$ i.e. $(x^1, \ldots, x^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ and $p^i = x^i, q^i = x^{n+i}$, the symplectic structure is

$$\omega^2 = dp \wedge dq = \sum_{i=1}^{n} dp^i \wedge dq^i = \sum_{i=1}^{n} dx^i \wedge dq^{n+i}. \tag{24}$$

Theorem *ω^2 satisfies a closed condition $d\,\omega^2 = 0$, if and only if there exists*

$$\frac{\partial \omega_{ij}}{\partial x_k} + \frac{\partial \omega_{jk}}{\partial x_i} + \frac{\partial \omega_{ki}}{\partial x_j} = 0 \quad (i \neq j \neq k). \tag{25}$$

The Hamiltonian system that is defined by the equations

$$\frac{dq_i}{dt} = -\frac{\partial H^i(q,p)}{\partial q^i}, \qquad \frac{dp_i}{dt} = \frac{\partial H^i(q,p)}{\partial p^i}, \tag{26}$$

which can be realized in the following form [4]. *A Hamiltonian system* (26) *on the phase space* R^{2n} *has the differential* 1*-form*

$$\beta = -\frac{\partial H}{\partial q^i} dp_i + \frac{\partial H}{\partial p^i} dq^i \tag{27}$$

and satisfies a closed form $d\beta = 0$, *where* d *is the exterior derivative. The exterior derivative for the phase space is defined as*

$$d = dq_i \frac{\partial}{\partial q_i} + dp_i \frac{\partial}{\partial p_i}. \tag{28}$$

Here and later, we mean the sum on the repeated indices i *and* j *from* 1 *to* n.
In symplectic space $(R^{2n}, dp \wedge dq)$, *the Poisson bracket is defined as follows*

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right). \tag{29}$$

The Hamiltonian systems can be realized in the following form

$$\dot{p}^j = \{p^j, H\}, \qquad \dot{q}^j = \{q^j, H\}. \tag{30}$$

If the right-hand sides of (27) *satisfy the involutive conditions for the phase space, which have the following forms*:

$$\{H^i, H^j\} = 0. \tag{31}$$

A Hamiltonian system (26) *on the phase space* R^{2n} *has the differential* 1*-form*

$$\beta = -\frac{\partial H}{\partial q^i} dp_i + \frac{\partial H}{\partial p^i} dq^i \tag{32}$$

and it is an exact form $\beta = dH$, *where* d *is the exterior derivative and* $H = H(q,p)$ *is a continuous differentiable unique function on the phase space.*

5 Fractional Hamiltonian Systems

Fractional generalization of the differential form, which is used in the definition of the Hamiltonian system, can be defined in the following form:

$$\beta_\alpha = -\mathbf{D}^\alpha_{p_i} H (dp_i)^\alpha + \mathbf{D}^\alpha_{q_i} H (dq_i)^\alpha. \tag{33}$$

Let us consider the canonical coordinates $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) = (q^1, \ldots, q^n, \; p^1, \ldots, p^n)$ in the phase space R^{2n} and a Hamiltonian system that is defined by the equations

$$\frac{dq_i}{dt} = -\mathbf{D}^\alpha_{p_i} H(q,p), \qquad \frac{dp_i}{dt} = \mathbf{D}^\alpha_{q_i} H(q,p). \tag{34}$$

The fractional generalization of Hamiltonian systems can be defined by using fractional generalization of differential forms.

Definition 1 *A Hamiltonian system* (26) *on the phase space* R^{2n} *is called a fractional Hamiltonian system if it has the fractional differential* 1*-form*

$$\beta_\alpha = -\mathbf{D}^\alpha_{p_i} H (dp_i)^\alpha + \mathbf{D}^\alpha_{q_i} H (dq_i)^\alpha.$$

Definition 2 *The fractional exterior derivative for the phase space* R^{2n} *is defined as*

$$d^\alpha = (dq_i)^\alpha \mathbf{D}^\alpha_{q_i} + (dp_i)^\alpha \mathbf{D}^\alpha_{p_i}. \tag{35}$$

For example, the fractional exterior derivative of order α *of* q^k, *with the initial point taken to be zero and* $n = 2$, *is given by*

$$d^\alpha q^k = (dq)^\alpha \mathbf{D}^\alpha_q q^k + (dp)^\alpha \mathbf{D}^\alpha_p q^k. \tag{36}$$

Using (8) *and* (36), *we have the following relation for the fractional exterior derivative*:

$$d^\alpha q^k = (dq)^\alpha \frac{\Gamma(k+1) q^{k-\alpha}}{\Gamma(k+1-\alpha)} + (dp)^\alpha \frac{q^k p^{-\alpha}}{\Gamma(1-\alpha)}. \tag{37}$$

Definition 3 *In symplectic space* $(R^{2n}, dp \wedge dq)$, *the fractional Poisson bracket is defined as follows*

$$\{F, H\}_\alpha = (\mathbf{D}^\alpha_{q_i} F \mathbf{D}^\alpha_{p_i} H - \mathbf{D}^\alpha_{p_i} F \mathbf{D}^\alpha_{q_i} H). \tag{38}$$

Proposition 1 *A Hamiltonian system* (34) *has a closed fractional form*

$$d^\alpha \beta_\alpha = 0, \tag{39}$$

where d^α *is the fractional exterior derivative.*

Proof In the canonical coordinates (q, p), the vector fields that define the system have the components $(-\mathbf{D}^\alpha_{p_i} H, \mathbf{D}^\alpha_{q_i} H)$, which are sued in (33). The 1-form β_α is defined by the equation

$$\beta_\alpha = -\mathbf{D}^\alpha_{p_i} H (dp_i)^\alpha + \mathbf{D}^\alpha_{q_i} H (dq_i)^\alpha. \tag{40}$$

The exterior derivative for this form can now be given by the relation

$$d^\alpha \beta_\alpha = d^\alpha (-\mathbf{D}^\alpha_{p_i} H (dp_i)^\alpha + \mathbf{D}^\alpha_{q_i} H (dq_i)^\alpha). \tag{41}$$

Using the rule

$$\mathbf{D}^\alpha_x (fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (\mathbf{D}^{\alpha-k}_x f) \frac{\partial^k g}{\partial x^k}$$

and the relation

$$\frac{\partial^k}{\partial x^k} (I dx_i)^\alpha) = 0 \quad (k \geq 1),$$

we get that

$$d^\alpha(A^i(dx_i)^\alpha) = \sum_{k=0}^{\infty}(dx_j)^\alpha \Lambda \binom{\alpha}{k} (\mathbf{D}^{\alpha-k}_{x_j} A^i) \frac{\partial^k}{\partial x_j^k}(dx_i)^\alpha = (dx_j)^\alpha \Lambda (dx_i)^\alpha \binom{\alpha}{0} (\mathbf{D}^\alpha_{x_j} A^i). \tag{42}$$

Here, we use

$$\binom{\alpha}{k} = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}.$$

Therefore, we have

$$\begin{aligned} d^\alpha \beta_\alpha = &-\mathbf{D}^\alpha_{q_j}\mathbf{D}^\alpha_{p_i} H(dq_j)^\alpha \Lambda (dp_i)^\alpha - \mathbf{D}^\alpha_{p_j}\mathbf{D}^\alpha_{p_i} H(dp_j)^\alpha \Lambda (dp_i)^\alpha \\ &+ \mathbf{D}^\alpha_{q_j}\mathbf{D}^\alpha_{q_i} H(dq_j)^\alpha \Lambda (dq_i)^\alpha + \mathbf{D}^\alpha_{p_j}\mathbf{D}^\alpha_{p_i} H(dq_j)^\alpha \Lambda (dq_i)^\alpha. \end{aligned} \tag{43}$$

This equation can be rewritten in an equivalent form

$$\begin{aligned} d^\alpha \beta_\alpha = &(-\mathbf{D}^\alpha_{q_i}\mathbf{D}^\alpha_{p_i} H^j + \mathbf{D}^\alpha_{p_i}\mathbf{D}^\alpha_{q_i} H^i)(dq_i)^\alpha \Lambda (dp_j)^\alpha \\ &+ \frac{1}{2}(-\mathbf{D}^\alpha_{p_i}\mathbf{D}^\alpha_{p_i} H^j + \mathbf{D}^\alpha_{p_j}\mathbf{D}^\alpha_{p_i} H^i)(dp_i)^\alpha \Lambda (dp_j)^\alpha \\ &+ \frac{1}{2}(\mathbf{D}^\alpha_{q_j}\mathbf{D}^\alpha_{q_i} H^i - \mathbf{D}^\alpha_{q_i}\mathbf{D}^\alpha_{q_i} H^j)(dq_i)^\alpha \Lambda (dq_j)^\alpha. \end{aligned} \tag{44}$$

Here, we use the Poisson bracket. It is obvious that the equation $d^\alpha \beta_\alpha = 0$, i.e., β_α is a closed fractional form. □

Proposition 2 *A Hamiltonian system* (34) *on the phase space* R^{2n} *is a fractional Hamiltonian system that is defined by the Hamiltonian* $H = H(q, p)$ *if the fractional differential* 1*-form*

$$\beta_\alpha = -\mathbf{D}^\alpha_{p_i} H^i (dp_i)^\alpha + \mathbf{D}^\alpha_{q_i} H^i (dq_i)^\alpha$$

is an exact fractional form

$$\beta_\alpha = d^\alpha H, \tag{45}$$

where d^α *is the fractional exterior derivative and* $H = H(q, p)$ *is a continuous differentiable function on the phase space.*

Proof Suppose that the fractional differential 1-form β_α, which is defined by (40), has the form

$$\beta_\alpha = d^\alpha H = (dp_i)^\alpha \mathbf{D}^\alpha_{p_i} H + (dq_i)^\alpha \mathbf{D}^\alpha_{q_i} H.$$

Therefore, the equations of motion for fractional Hamiltonian systems can be written in the form

$$\frac{dq_i}{dt} = \mathbf{D}^\alpha_{p_i} H, \qquad \frac{dp_i}{dt} = -\mathbf{D}^\alpha_{q_i} H. \tag{46}$$

The fractional differential 1-form β_α for the fractional Hamiltonian system with Hamiltonian H can be written in the form $\beta_\alpha = d^\alpha H$. If the exact fractional differential 1-form β_α is equal to zero ($d^\alpha H = 0$), then we can get the equation that defines the stationary states of the Hamiltonian system. □

In the phase space R^{2n} and a fractional Hamiltonian systems can be defined by the equations

$$\mathbf{D}_t^\alpha \begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{p_i}^\alpha & 0 \\ 0 & -\mathbf{D}_{q_i}^\alpha \end{pmatrix} \begin{pmatrix} H_n \\ H_n \end{pmatrix} = J \begin{pmatrix} H_n \\ H_n \end{pmatrix}. \tag{47}$$

The fractional generalization of Hamiltonian system can be defined by the equations generalization of differential forms [14].

A generalized Hamiltonian system (47) on the phase space R^{2n} it called a fractional Hamiltonian system if the fractional differential 1-form $\beta_\alpha = -\mathbf{D}_{p_i}^\alpha H^i (dp_i)^\alpha + \mathbf{D}_{q_i}^\alpha H^i (dq_i)^\alpha$ is a closed fractional form $d^\alpha \beta_\alpha = 0$, where d^α is the fractional exterior derivative.

A central and very important subject in the theory of integrable system is to search for a sequence of scalar functions $\{H_n\}$ such that (46) can be cast in the Hamiltonian form (47). Equation (47) is the fractional form Hamiltonian structure of soliton equation hierarchy, which is different from the result in [16].

6 The Generalized Fractional AKNS Equation Hierarchy and Hamiltonian System

To illustrate our method, we want to apply the fractional zero curvature equation to construct the generalized fractional AKNS equation hierarchy. We consider the following matrix spectral problem

$$\phi_x = U(u,\lambda)\phi, \qquad U(u,\lambda) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \qquad \mathbf{D}_{t_n}^\alpha \lambda = 0, \tag{48}$$

where λ is a spectral parameter. The spectral problem (48) is called AKNS spectral problem [19].

To derive an associated soliton hierarchy, we first solve the adjoint equation

$$W_x = [U, W] \tag{49}$$

of the spectral problem (48) through the generalized Tu scheme [20]. We assume that a solution W is given by

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{50}$$

Therefore, the adjoint (49) is equivalent to

$$\begin{cases} a_x = qc - rb, \\ b_x = -2\lambda b - 2qa, \\ c_x = 2\lambda c + 2ra. \end{cases} \tag{51}$$

Let us seek a formal solution of the type

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & -a^{(k)} \end{pmatrix} \lambda^{-k}, \tag{52}$$

then, the condition (51) becomes the following recursion relation:

$$\begin{cases} a^{(0)} = -1, \quad b^{(0)} = 0, \quad c^{(0)} = 0, \\ a^{(1)} = 0, \quad b^{(1)} = q, \quad c^{(1)} = r, \\ a^{(2)} = \frac{1}{2}qr, \quad b^{(2)} = -\frac{1}{2}q_x, \quad c^{(2)} = \frac{1}{2}r_x, \\ a^{(3)} = \frac{1}{4}(qr)_x, \quad b^{(3)} = \frac{1}{4}(q_{xx} - 2q^2 r), \quad c^{(3)} = \frac{1}{4}(r_{xx} - 2r^2 q), \\ a_x^{(n)} = qc^{(n)} - rb^{(n)}, \\ b^{(n+1)} = -\frac{1}{2}(b_x^{(n)} - 2qa^{(n)}), \\ c^{(n+1)} = \frac{1}{2}(c_x^{(n)} - 2ra^{(n)}). \end{cases} \tag{53}$$

In the compatibility condition, i.e., the generalized zero curvature equation

$$\mathbf{D}_{t_n}^{\alpha} U - \mathbf{D}_x^{\alpha} V^{(n)} + [U, V^{(n)}] = 0,$$

we choose

$$W = \begin{pmatrix} \frac{1}{2}qr & -\frac{1}{2}q_x \\ \frac{1}{2}r_x & -\frac{1}{2}qr \end{pmatrix}. \tag{54}$$

Then we have

$$[U, W] = \begin{pmatrix} \frac{1}{2}qr_x + \frac{1}{2}q_x r & q^2 r + \lambda q_x \\ qr^2 + \lambda r_x & -\frac{1}{2}qr_x - \frac{1}{2}q_x r \end{pmatrix}. \tag{55}$$

Comparing the coefficient of the λ in (22), we obtain a system of evolution equations

$$\mathbf{D}_{t_n}^{\alpha} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} -\mathbf{D}_x^{\alpha} q \\ \mathbf{D}_x^{\alpha} r \end{pmatrix}, \tag{56}$$

which is the generalized fractional version of the AKNS nonlinear equation hierarchy.

$$\begin{cases} \mathbf{D}_{t_n}^{\alpha} q + \frac{1}{2}\mathbf{D}_x^{\alpha} q_x + q^2 r = 0, \\ \mathbf{D}_{t_n}^{\alpha} r - \frac{1}{2}\mathbf{D}_x^{\alpha} r_x + qr^2 = 0, \end{cases} \tag{57}$$

which is fractional Shrödinger equations. We obtained the Hamiltonian structure of the fractional AKNS hierarchy (56)

$$H = -(qr), \tag{58}$$

where the Hamiltonian operator J is defined by

$$J = \begin{pmatrix} 0 & \frac{1}{2}\mathbf{D}_x^{\alpha} \\ -\frac{1}{2}\mathbf{D}_x^{\alpha} & 0 \end{pmatrix}. \tag{59}$$

If we choose

$$W_1 = \begin{pmatrix} \frac{1}{4}(qr)_x & \frac{1}{4}(q_{xx} - 2q^2 r) \\ \frac{1}{4}(r_{xx} - 2r^2 q) & -\frac{1}{4}(qr)_x \end{pmatrix},$$

then we have

$$[U, W_1] = \begin{pmatrix} \frac{1}{4}(qr_{xx} - rq_{xx}) & -\frac{1}{2}(q_{xx} - 2q^2 r)\lambda - \frac{1}{2}q(qr)_x \\ \frac{1}{2}(r_{xx} - 2r^2 q)\lambda + \frac{1}{2}r(qr)_x & \frac{1}{4}(rq_{xx} - qr_{xx}) \end{pmatrix}.$$

Comparing the coefficient of the λ in (22), we obtain a system of evolution equations

$$\begin{cases} \mathbf{D}_{t_n}^{\alpha} q - \frac{1}{4}\mathbf{D}_x^{\alpha}(q_{xx} - 2q^2 r) - \frac{1}{2} q(qr)_x = 0, \\ \mathbf{D}_{t_n}^{\alpha} r - \frac{1}{4}\mathbf{D}_x^{\alpha}(r_{xx} - 2r^2 q) + \frac{1}{2} r(qr)_x = 0, \end{cases}$$

which are new fractional nonlinear equations.

Fractional derivatives and integrals have found many applications in recent studies in physics. The interest in fractional analysis has been growing continually during the past few years. Using the fractional derivatives and fractional differential forms, we consider the fractional generalized Hamiltonian systems. In the general case, the fractional Hamiltonian systems of AKNS equation hierarchy is obtained.

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